On some properties of q-Hahn multiple orthogonal polynomials

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Abstract

This contribution deals with multiple orthogonal polynomials of type II with respect to q-discrete measures (q-Hahn measures). In addition, we show that this family of multiple orthogonal polynomials has a lowering operator, and raising operators as well as a Rodrigues type formula. The combination of lowering and raising operators leads to a third order q-difference equation when two orthogonality conditions are considered. An explicit expression of this q-difference equation will be given. Indeed, this q-difference equation relates polynomial with a given degree evaluated at four consecutive non-uniformed distributed points, which makes these polynomials interesting from the point of view of bispectral problems.

Key words: Multiple orthogonal polynomials; Hermite-Padé approximation; Difference equations; q-polynomials; Hahn polynomials

Dedicated to Professor Jesús S. Dehesa on the occasion of his 60th birthday

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1 Introduction

1.1 Orthogonal polynomials of a discrete variable

A polynomial sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to a positive measure μ on the real line is such that P_n has degree n and satisfies the conditions

$$\int_\Omega P_n(x)h(x)\,d\mu(x) = 0, \quad \forall h(x) \in \mathbb{P}_k, \quad k = 0, 1, \dots, n-1, \quad \Omega \subset \mathbb{R}, \quad (1)$$

where $\mathbb{P}_k$ denotes the vector space of polynomials of degree at most k [6]. Thus, $P_n(x)$ is defined up to a multiplicative factor. In the case of discrete orthogonal polynomials, we have a discrete measure μ (with finite moments)

$$\mu = \sum_{k=0}^{N} \omega_k \delta_{x_k}, \quad \omega_k > 0, \; x_k \in \mathbb{R} \text{ and } N \in \mathbb{N} \cup \{+\infty\},$$

which is a linear combination of Dirac measures on the $N+1$ points $x_0, \dots, x_N$. The orthogonality conditions of a discrete orthogonal polynomial P_n on the lattice $\{x(s) \mapsto \mathbb{R}^+ : \; s = 0, 1, \dots, N\}$ are usually written as

$$\sum_{s=0}^{N} P_n(x(s)) x^k(s)\,\omega(s) \bigtriangleup x(s + \tfrac{1}{2}) = 0, \quad k = 0, 1, \dots, n-1,$$

where $\bigtriangleup x(s) = x(s+1) - x(s)$ is the forward difference operator.

If $x(s) = c_1 q^s + c_2 q^{-s} + c_3$, $(q \in \mathbb{R}^+ \backslash \{1\})$ or $x(s) = as^2 + bs + c$, where c_1, c_2, c_3, a, b, c are constants, and $\omega(s)$ is solution of the Pearson-type difference equation, the above polynomial sequence $\{P_n\}_{n\geq 0}$ is called classical orthogonal polynomials of a discrete variable [9] (see also [7]). There exist several mathematical approaches to the study of these polynomials. One of them considers $\{P_n\}_{n\geq 0}$ as polynomial solutions of the hypergeometric-type difference equation which can be interpreted as a discretization of the very well known hypergeometric differential equation (see [9])

$$\tilde{\sigma}(x)y''(x) + \tilde{\tau}(x)y'(x) + \lambda y(x) = 0, \quad \text{with } \deg \tilde{\sigma} \leq 2, \quad \deg \tilde{\tau} = 1, \quad \lambda = const.$$

This equation can be also obtained by combining the corresponding lowering and raising operators for each sequence of classical orthogonal polynomials (1). Similarly, for classical orthogonal polynomials of a discrete variable the corresponding lowering and raising operators leads to the hypergeometric-type difference equation

$$
\begin{aligned}
&\sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)}+\tau(s)\frac{\triangle y(s)}{\triangle x(s)}+\lambda y(s)=0,\\
&\sigma(s)=\tilde{\sigma}(x(s))-\tfrac{1}{2}\tilde{\tau}(x(s))\triangle x(s-\tfrac{1}{2}),\quad \tau(s)=\tilde{\tau}(x(s)),
\end{aligned}
\tag{2}
$$

being $\nabla y(s)=\triangle y(s-1)$ the backward difference operator. In the above equation for the specific choice [3]

$$
\begin{aligned}
&\sigma(s)=-q^{-\frac{N+\alpha}{2}}x(s)^2+q^{-\frac{1}{2}}[N+\alpha]_q x(s),\quad \text{where } [x]_q\stackrel{\text{def}}{=}\frac{q^{\frac{x}{2}}-q^{-\frac{x}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}},\\
&\tau(s)=-q^{\frac{\beta+2-N}{2}}[\alpha+\beta+2]_q x(s)+q^{\alpha+\beta+1}[\beta+1]_q[N-1]_q,\quad N\in\mathbb{N},
\end{aligned}
$$

the solution of the Person-type equation $\dfrac{\triangle}{\triangle x(s-\frac{1}{2})}\left[\sigma(s)\omega(s)\right]=\tau(s)\omega(s)$ is given by

$$
\omega(s)=q^{(\frac{\alpha+\beta}{2})s}\frac{\tilde{\Gamma}_q(s+\beta+1)\tilde{\Gamma}_q(N+\alpha-s)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s)},\quad \text{where } \tilde{\Gamma}(s)=q^{-\frac{(s-1)(s-2)}{4}}\Gamma_q(s),
$$

and

$$
\Gamma_q(s)=\begin{cases}
f(s;q)=(1-q)^{1-s}\dfrac{\prod\limits_{k\geq 0}(1-q^{k+1})}{\prod\limits_{k\geq 0}(1-q^{s+k})}, & 0<q<1,\\
q^{\frac{(s-1)(s-2)}{2}}f(s;q^{-1}), & q>1.
\end{cases}
\tag{3}
$$

Notice that when q goes to 1, $\Gamma_q(s)$ tends to the Gamma function $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$, $s>0$, [1]. The weight function $\omega(s)$ is the so-called symmetrization factor of (2), which allows to write (2) in the self-adjoint form.

All the characteristics and main properties of the corresponding q-Hahn polynomials as solution of the hypergeometric-type difference equation on the lattice $x(s)=\frac{q^s-1}{q-1}$ have been computed in [3].

1.2 Multiple orthogonal polynomials

The main ingredients for constructing multiple orthogonal polynomials are the set of measures $\mu_1,\mu_2,\ldots,\mu_r$ ($r\geq 2$) on $\mathbb{R}$, and the multi-index $\vec{n}=(n_1,n_2,\ldots,n_r)\in\mathbb{N}^r$ and its length $|\vec{n}|=n_1+n_2+\cdots+n_r$ (see [2], [5], [11]). A type II multiple orthogonal polynomial $P_{\vec{n}}(x)$ of degree $\leq|\vec{n}|$ is defined by the orthogonality conditions

$$
\begin{aligned}
&\int_{\Omega_1} P_{\vec{n}}(x)x^k \, d\mu_1(x) = 0, \ k = 0, 1, \ldots, n_1 - 1, \\
&\qquad\qquad\qquad\vdots \\
&\int_{\Omega_r} P_{\vec{n}}(x)x^k \, d\mu_r(x) = 0, \ k = 0, 1, \ldots, n_r - 1,
\end{aligned} \tag{4}
$$

where Ω_i is the smallest interval that contains $\operatorname{supp}(\mu_i)$, $i = 1, 2, \ldots, r$. Conditions (4) give a linear system of $|\vec{n}|$ homogeneous equations for the $|\vec{n}| + 1$ unknown coefficients of $P_{\vec{n}}(x)$. If the multi-index $\vec{n}$ is *normal* [10] the solution is a unique polynomial $P_{\vec{n}}(x)$ (up to a multiplicative factor) of degree exactly $|\vec{n}|$. In this situation throughout the paper we consider always monic multiple orthogonal polynomials.

If the measures in (4) are positive discrete measures on $\mathbb{R}$, i.e.,

$$
\mu_i = \sum_{k=0}^{N_i} \omega_{i,k} \delta_{x_{i,k}}, \qquad \omega_{i,k} > 0, \ x_{i,k} \in \mathbb{R}, \quad N_i \in \mathbb{N} \cup \{+\infty\}, \quad i = 1, \ldots, r,
$$

where $x_{i_1,k} \neq x_{i_2,k}$, $k = 0, \ldots, N_i$, whenever $i_1 \neq i_2$, the corresponding polynomial solution is then a *discrete multiple orthogonal polynomial* $P_{\vec{n}}(x)$. Here we have that $\operatorname{supp}(\mu_i)$ is the closure of $\{x_{i,k}\}_{k=0}^{N_i}$ and that Ω_i is the smallest closed interval on $\mathbb{R}$ which contains $\{x_{i,k}\}_{k=0}^{N_i}$. If the above system of measures forms an *AT system* [10] then every multi-index is normal.

Definition 1.1 *Let $\mu_1, \mu_2, \ldots, \mu_r$ be a system of positive discrete measures*

$$
\mu_i = \sum_{k=0}^{N} \omega_{i,k} \delta_{x_k}, \qquad \omega_{i,k} > 0, \ x_k \in \mathbb{R}, \quad N \in \mathbb{N} \cup \{+\infty\}, \quad i = 1, \ldots, r,
$$

so that supp(μ_i) is the closure of $\{x_k\}_{k=0}^{N}$ and $\Omega_i = \Omega$ for each $i = 1, \ldots, r$.

The system $\mu_1, \mu_2, \ldots, \mu_r$ is an AT system if there exist r continuous functions $v_1, \ldots, v_r$ on Ω with $v_i(x_k) = \omega_{i,k}$, $k = 1, \ldots, N$, $i = 1, \ldots, r$, such that the $|\vec{n}|$ functions

$$
v_1(x), xv_1(x), \ldots, x^{n_1-1}v_1(x), \ldots, v_r(x), xv_r(x), \ldots, x^{n_r-1}v_r(x),
$$

form a Chebyshev system on Ω for each multi-index $\vec{n}$ with $|\vec{n}| < N + 1$, i.e., every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x)v_i(x)$, where $Q_{n_i-1} \in \mathbb{P}_{n_i-1} \setminus \{0\}$, has at most $|\vec{n}| - 1$ zeros on Ω.

In [5] for several AT system of measures was studied the corresponding discrete multiple orthogonal polynomials of type II on the linear lattice $x(s) = s$ (those of Charlier, Kravchuk, Meixner of first and second kind, and Hahn). Even more, it was obtained rising operators and then the Rodrigues-type formula when r orthogonality conditions are considered.

The monic multiple Hahn polynomials corresponding to the multi-index $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and the parameters $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$, α_0, and N, is the unique monic polynomial $H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)$ of degree $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ which satisfies the conditions

$$\begin{aligned}
&\sum_{s=0}^{N} H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)(-s)_i v^{\alpha_1,\alpha_0,N}(s) = 0, \qquad i = 0, \ldots, n_1 - 1,\\
&\qquad\qquad\vdots\\
&\sum_{k=0}^{N} H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)(-s)_i v^{\alpha_r,\alpha_0,N}(s) = 0, \qquad i = 0, \ldots, n_r - 1,
\end{aligned}$$

with

$$v^{\alpha_i,\alpha_0,N}(s) = \begin{cases} \dfrac{\Gamma(\alpha_i + s + 1)}{\Gamma(s+1)} \dfrac{\Gamma(\alpha_0 + N - s + 1)}{\Gamma(N - s + 1)}, & \text{if } s = 0, 1, \ldots, N \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Notice that $v^{\alpha_i,\alpha_0,N}(s)$ is a C^∞-function on $\mathbb{R} \setminus (\{-\alpha-1, -\alpha-2, \ldots\} \cup \{\alpha_0 + N + 1, \alpha_0 + N + 2, \alpha_0 + N + 3, \ldots\})$ with simple poles at the points of $\{-\alpha - 1, -\alpha - 2, \ldots\} \cup \{\alpha_0 + N + 1, \alpha_0 + N + 2, \alpha_0 + N + 3, \ldots\}$. In [5] we proved the normality of the multi-index $\vec{n} = (n_1, \ldots, n_r)$ with $|\vec{n}| < N + 1$, whenever $\alpha_i - \alpha_j \notin \{0, 1, \ldots, N-1\}$ when $i \neq j$. Also was found the following raising operators:

$$\begin{aligned}
&\left(\frac{1}{v^{\alpha_i - 1,\alpha_0 - 1,N+1}(s)} \nabla\, v^{\alpha_i,\alpha_0,N}(s) \right) H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(s) \\
&= -(|\vec{n}| + \alpha_i + \alpha_0) H_{\vec{n}+\vec{e}_i}^{\vec{\alpha}-\vec{e}_i,\alpha_0-1,N+1}(s), \qquad i = 1, \ldots, r.
\end{aligned} \tag{6}$$

Here the multi-index $\vec{e}_i$ is the standard r dimensional unit vector with the ith entry equals 1 and 0 otherwise.

As a consequence of (6) the *Rodrigues-type formula*

$$\begin{aligned}
H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(s) &= \frac{(-1)^{|\vec{n}|}}{\prod_{k=1}^{r} (|\vec{n}| + \alpha_k + \alpha_0 + 1)_{n_k}} \frac{\Gamma(s+1)\Gamma(N-s+1)}{\Gamma(\alpha_0 + N - s + 1)} \\
\times \prod_{i=1}^{r} &\left(\frac{1}{\Gamma(\alpha_i + s + 1)} \nabla^{n_i}\, \Gamma(\alpha_i + n_i + s + 1) \right) \frac{\Gamma(\alpha_0 + N - s + 1)}{\Gamma(s+1)\Gamma(N - |\vec{n}| - s + 1)},
\end{aligned} \tag{7}$$

yields. This property enables us to find an explicit expression for the polynomials in the case $r = 2$ (see [5]).

In [8] the author found the lowering operators for the multiple orthogonal polynomials studied in [5]. A combination of lowering and rising operators (6) leads to $(r+1)$th order difference equations having discrete multiple orthogonal polynomials as solutions (see [8]). In particular, the polynomial coefficients for the 3-order difference equation having $H_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)$ as eigenfunction were found. Indeed,

$$\sigma_3(s) \bigtriangleup \bigtriangledown^2 y + \sigma_2(s) \bigtriangleup \bigtriangledown y + \sigma_1(s) \bigtriangleup y + \tau_1(s) \bigtriangledown y + \tau_0(s) y = 0, \qquad (8)$$

being

$$\begin{aligned}
\sigma_3(s) &= s(s-1)(\alpha_0+N-s+1)(\alpha_0+N-s+2),\\
\sigma_2(s) &= s(\alpha_0+N-s+1)\left[(\alpha_1+\alpha_2+3)(N-s+1)+2(s-1)(1-\alpha_0)\right],\\
\sigma_1(s) &= \left[(\alpha_1+1)(\alpha_2+1)(N-s)-(\alpha_1+\alpha_2+3)(\alpha_0+1)s\right](N-s+1)\\
&\quad +\alpha_0(\alpha_0+1)s(s-1),\\
\tau_1(s) &= \left[(n_1+n_2)(n_1+n_2+\alpha_0+1)+n_1\alpha_1+n_2\alpha_2-n_1n_2\right]\\
&\quad \times s(\alpha_0+N-s+1),\\
\tau_2(s) &= \left[(n_1+n_2)(n_1+n_2+\alpha_0+1)+n_1\alpha_1+n_2\alpha_2-n_1n_2\right]\\
&\quad \times(N+1-(\alpha_0+1)s).
\end{aligned}$$

In this paper we will obtain a q-difference equation of order 3 on the q-exponential lattice $x(s) = \frac{q^s-1}{q-1}$ having q-multiple orthogonal polynomials as solution, i.e., a q-extension of equation (8). This q-difference equation leads to a relation for the polynomial with a given degree evaluated at non-uniformed distributed points $x(s)$, $x(s+1)$, $x(s+2)$ and $x(s+3)$. Thus, the q-multiple orthogonal polynomials studied here are solution of a bispectral problem since they satisfy both a recurrence relation in the variable $x(s)$ and a recurrence relation in the degree [4].

2 q-Multiple orthogonal polynomials: q-Hahn case

Definition 2.1 *A polynomial $P_{\vec{n}}(x(s))$ on the lattice $x(s) = \frac{q^s-1}{q-1}$, $q \in (0,1)$ is said to be a q-multiple orthogonal polynomial of a multi-index $\vec{n} \in \mathbb{N}^r$ with respect to positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$ such that* $\operatorname{supp} \mu_i \subset \Omega_i \subset \mathbb{R}$, *$i = 1, 2, \ldots, r$, if the conditions*

$$\begin{aligned}
&(a) \qquad \deg P_{\vec{n}}(x(s)) \leq |\vec{n}| = n_1+n_2+\cdots+n_r,\\
&(b) \ \sum_{i=1}^{N_i} P_{\vec{n}}(x(s))x(s)^k d\mu_i = 0, \quad k = 0,1,\ldots,n_i-1, \quad i = 1,2,\ldots,r,
\end{aligned} \qquad (9)$$

hold.

The q-multiple orthogonal polynomials of Hahn are constructed by considering specifically the following r positive discrete measures on $\mathbb{R}$

$$\mu_i = \sum_{s=0}^{N} \omega_i(k)\delta(k-s), \quad \omega_i > 0, \quad N \in \mathbb{N}, \quad i = 1, \dots, r,$$

where $\omega_i(s) = v_q^{\alpha_i,\alpha_0,N}(s) \triangle x(s-\frac{1}{2})$, and

$$v_q^{\alpha_i,\alpha_0,N}(s) = \begin{cases} q^{\frac{\alpha_i+\alpha_0}{2}s} \dfrac{\tilde{\Gamma}_q(s+\alpha_i+1)\tilde{\Gamma}_q(N+\alpha_0-s+1)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s+1)}, & \text{if } s = 0, 1, \dots, N \\ 0, & \text{otherwise,} \end{cases}$$

being $\alpha_0, \alpha_i > -1$, $\alpha_i - \alpha_j \notin \{0, 1, \dots, N-1\}$ when $i \neq j$.

According with definition 2.1 the support of the measures is the closure of $\{x(s)\}_{s=0}^N$ and Ω represents the smallest closed interval on $\mathbb{R}$ that contains $\{x(s)\}_{s=0}^N$ since $N_1 = N_2 = \cdots = N_r = N$.

Lemma 2.1 *There exists a number $q_0 \in (0,1)$ such that $\forall q \in (q_0, 1)$ the measures $\mu_1, \mu_2, \dots, \mu_r$ forms an AT system on $\mathbb{R}^+$.*

Proof. $v_q^{\alpha_i,\alpha_0,N}(s)$ is a C^∞-function on $\mathbb{R} \setminus (\{-\alpha-1, -\alpha-2, \dots\} \cup \{\alpha_0 + N+1, \alpha_0+N+2, \alpha_0+N+3, \dots\})$ with simple poles at the points of $\{-\alpha-1, -\alpha-2, \dots\} \cup \{\alpha_0+N+1, \alpha_0+N+2, \alpha_0+N+3, \dots\}$. The multi-index $\vec{n} = (n_1, \dots, n_r)$ with $|\vec{n}| < N+1$ is normal for the Hahn system of weights (5) whenever $\alpha_i - \alpha_j \notin \{0, 1, \dots, N-1\}$ when $i \neq j$ (see [5]). Here we has this when q tends to 1. Therefore, $\lim_{q\to 1} \sum_{i=1}^r Q_{n_i-1}(x(s)) v_q^{\alpha_i,\alpha_0,N}(s)$ has at most $|\vec{n}|-1$ zeros on $\mathbb{R}^+$ for every $Q_{n_i-1}(x(s)) \in \mathbb{P}_{n_i-1} \setminus \{0\}$. Thus, the $|\vec{n}|$ functions

$$\begin{gathered} v_q^{\alpha_1,\alpha_0,N}(s), x(s)v_q^{\alpha_1,\alpha_0,N}(s), \dots, x(s)^{n_1-1}v_q^{\alpha_1,\alpha_0,N}(s), \dots, \\ v_q^{\alpha_r,\alpha_0,N}(s), x(s)v_q^{\alpha_r,\alpha_0,N}(s), \dots, x(s)^{n_r-1}v_q^{\alpha_r,\alpha_0,N}(s), \end{gathered}$$

form a Chebyshev system on $\mathbb{R}^+$ for $q \in (q_0, 1)$. $\square$

Next we will assume that q belongs to the interval $(q_0, 1)$, however expression (12) and consequently the Rodrigues type formula are formally valid for $q \in (0, 1)$.

The monic q-Hahn multiple orthogonal polynomial [4] corresponding to the multi-index $\vec{n} = (n_1, \dots, n_r)$ and the set of parameters N, α_0 and $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ is the unique monic polynomial $P_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x(s))$ of degree $|\vec{n}|$ which satisfies the conditions

$$\begin{cases} \sum_{s=0}^{N} P_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x(s))h(x(s))v_q^{\alpha_i,\alpha_0,N}(s) \bigtriangledown x(s+\tfrac{1}{2}) = 0, \\ \qquad k = 0,1,\ldots,n_i-1, \quad i = 1,2,\ldots,r, \end{cases} \tag{10}$$

where $h(x(s))$ belong to the vector space of polynomials $\mathbb{P}_k$ in the variable $x(s)$. The q-orthogonality conditions are more conveniently expressed when $h(x(s))$ is the q-analog of the Stirling polynomials

$$(s)_q^{[k]} = \prod_{j=0}^{k-1} \frac{q^{s-j}-1}{q-1} = x(s)x(s-1)\cdots x(s-k+1).$$

Notice that when q goes to 1, $(s)_q^{[k]}$ converges to $(-1)^k(-s)_k$, where $(s)_k$ denotes the Pochhammer symbol $(s)_k = s(s+1)\cdots(s+k-1)$, $(s)_0 = 1$. From now on we will denote any polynomial $P_n(x(s))$ simply as $P_n(s)$.

In [4] for q-multiple Hahn polynomials we obtained r raising operators

$$D^{\alpha_i,\alpha_0,N}\, P_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(s) = -q^{-\frac{N+|\vec{n}|+\alpha_0}{2}}\, [|\vec{n}|+\alpha_i+\alpha_0]_q\, P_{\vec{n}+\vec{e}_i}^{\vec{\alpha}-\vec{e}_i,\alpha_0-1,N+1}(s), \tag{11}$$

where $D^{\alpha_i,\alpha_0,N} f(s) \stackrel{\text{def}}{=} \dfrac{1}{v_q^{\alpha_i-1,\alpha_0-1,N+1}(s)} \nabla v_q^{\alpha_i,\alpha_0,N}(s) f(s)$, and $\nabla \stackrel{\text{def}}{=} \dfrac{\bigtriangledown}{\bigtriangledown x(s)}$, for any continuous function f, $i = 1,2,\ldots,r$. Indeed,

$$\begin{aligned} D^{\alpha_i,\alpha_0,N} f(s) = &\left([s+\alpha_i]_q[N-s+1]_q - \frac{[s]_q[N+\alpha_0-s+1]_q}{q^{\frac{\alpha_i+\alpha_0}{2}}} \right) f(s) \\ &+ \frac{[s]_q[N+\alpha_0-s+1]_q}{q^{\frac{\alpha_i+\alpha_0}{2}}} \bigtriangledown f(s). \end{aligned} \tag{12}$$

Notice that we call $D^{\alpha_i,\alpha_0,N}$ a raising operator since the ith component of the multi-index $\vec{n}$ in (11) is increased by 1.

In this paper we will consider without loss of generality $r = 2$. In [4] was considered the general case. Thus, as a consequence of (11) the Rodrigues-type formula

$$\begin{aligned} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s) = c_q^{\vec{n},\vec{\alpha},\alpha_0,N} \frac{q^{-\frac{\alpha_0}{2}s}\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s+1)}{\tilde{\Gamma}_q(\alpha_0+N-s+1)} \\ \times \left(\prod_{i=1}^{2} \frac{q^{-\frac{\alpha_i}{2}s}}{\tilde{\Gamma}_q(\alpha_i+s+1)} \nabla^{n_i} \frac{\tilde{\Gamma}_q(\alpha_i+n_i+s+1)}{q^{-\frac{\alpha_i+n_i}{2}s}} \right) \frac{q^{\frac{\alpha_0+|\vec{n}|}{2}s}\tilde{\Gamma}_q(\alpha_0+N-s+1)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-|\vec{n}|-s+1)}, \end{aligned}$$

where $c_q^{\vec{n},\vec{\alpha},\alpha_0,N} = \dfrac{(-1)^{n_1+n_2} q^{\frac{(N+\alpha_0)(n_1+n_2)+n_1n_2}{2}+\binom{n_1}{2}+\binom{n_2}{2}}}{\prod_{k=0}^{n_1-1}[2n_1+n_2+\alpha_0+\alpha_1-k]_q \prod_{l=0}^{n_2-1}[2n_2+n_1+\alpha_0+\alpha_2-l]_q}$, yields.

Lemma 2.2 *The q-Hahn multiple orthogonal polynomials satisfy the following property*

$$\begin{aligned}
&\sum_{s=0}^{N} P_{\vec{n}-\vec{e}_l}^{\vec{\alpha}+\vec{e}_l,\alpha_0+1,N-1}(s)(s)_q^{[n_k-1]} \upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s) \triangle x(s-\tfrac{1}{2}) \\
&= a_{k,l} \sum_{s=0}^{N} P_{n_1-1,n_2-1}^{\alpha_1+1,\alpha_2+1,\alpha_0+2,N-2}(s)(s)_q^{[n_k-1]} \upsilon_q^{\alpha_k+1,\alpha_0+2,N-2}(s) \triangle x(s-\tfrac{1}{2}),
\end{aligned} \tag{13}$$

where $k,l=1,2$ *and*

$$a_{k,l} = \frac{[n_1+n_2+\alpha_0+\alpha_l+1]_q}{q^{-\frac{(N+n_1+n_2+n_k+\alpha_k+\alpha_0-1)}{2}}[\alpha_k-\alpha_l+n_k]_q} \left(\prod_{j=1}^{2} \frac{[\alpha_k-\alpha_j+n_k]_q}{[n_1+n_2+\alpha_0+\alpha_j+1]_q} \right).$$

Proof. By shifting conveniently the parameters involved in (11) and (12), respectively one has

$$\begin{aligned}
P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s) &= \frac{-q^{\frac{N+n_1+n_2+\alpha_0-1}{2}}}{[n_1+n_2+\alpha_i+\alpha_0]_q} D^{\alpha_i+1,\alpha_0+1,N-1} P_{\vec{n}-\vec{e}_i}^{\vec{\alpha}+\vec{e}_i,\alpha_0+1,N-1}(s) \\
= \frac{-q^{\frac{N+n_1+n_2+\alpha_0-1}{2}}}{[n_1+n_2+\alpha_i+\alpha_0]_q} &\left\{ \left([s+\alpha_i+1]_q[N-s]_q - \frac{[s]_q[N+\alpha_0-s+1]_q}{q^{\frac{\alpha_i+\alpha_0+2}{2}}} \right) \mathcal{I} \right. \\
&\left. + \frac{[s]_q[N+\alpha_0-s+1]_q}{q^{\frac{\alpha_i+\alpha_0+2}{2}}} \nabla \right\} P_{\vec{n}-\vec{e}_i}^{\vec{\alpha}+\vec{e}_i,\alpha_0+1,N-1}(s), \quad i=1,2,
\end{aligned}$$

where $\mathcal{I}$ denotes the identity operator. Thus, for $k=1,2$ we obtain

$$\begin{aligned}
\sum_{s=0}^{N} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)(s)_q^{[n_k-1]} \upsilon_q^{\alpha_k+1,\alpha_0,N}(s) \triangle x(s-\tfrac{1}{2}) &= \frac{q^{\frac{N+n_1+n_2+\alpha_0-1}{2}}}{[n_1+n_2+\alpha_i+\alpha_0]_q} \\
\times \sum_{s=0}^{N} (s)_q^{[n_k-1]} \upsilon_q^{\alpha_k+1,\alpha_0,N}(s) \triangle x(s-\tfrac{1}{2}) &\left\{ \frac{[s]_q[N+\alpha_0-s+1]_q}{q^{\frac{\alpha_i+\alpha_0+2}{2}}} \mathcal{I} \right. \\
-[s+\alpha_i+1]_q[N-s]_q\mathcal{I} - \frac{[s]_q[N+\alpha_0-s+1]_q}{q^{\frac{\alpha_i+\alpha_0+2}{2}}} &\left. \nabla \right\} P_{\vec{n}-\vec{e}_i}^{\vec{\alpha}+\vec{e}_i,\alpha_0+1,N-1}(s).
\end{aligned} \tag{14}$$

Since $[N-s]_q \upsilon_q^{\alpha_k+1,\alpha_0,N}(s) = q^{-\frac{s}{2}} \upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)$, and

$$\begin{aligned}
&(s+1)_q^{[n_k-1]}\, [N+\alpha_0-s]_q\, \upsilon_q^{\alpha_k+1,\alpha_0,N}(s+1) \triangle x(s+\tfrac{1}{2}) \\
&= q^{\frac{\alpha_k+\alpha_0+3}{2}} (s)_q^{[n_k-2]} [\alpha_k+s+2]_q \upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s) \triangle x(s-\tfrac{1}{2}),
\end{aligned}$$

by using summation by parts in the above expression (14) one gets

$$\sum_{s=0}^{N} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0,N}(s)\triangle x(s-\tfrac{1}{2}) = \frac{q^a[n_k+\alpha_k-\alpha_i]_q}{[n_1+n_2+\alpha_i+\alpha_0+1]_q} \times \sum_{s=0}^{N} P_{\vec{n}-\vec{e}_i}^{\vec{\alpha}+\vec{e}_i,\alpha_0+1,N-1}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2}), \tag{15}$$

where $a = \frac{N+|\vec{n}|+n_k+\alpha_k+\alpha_0}{2}$.

Then, by using inductively (15) the statement holds. □

Next, we will find a *lowering operator*. Notice that for the lattice $x(s) = \frac{q^s-1}{q-1}$ and any $k \in \mathbb{N}$, $\dfrac{\triangle x^k(s)}{\triangle x(s)}$ is a polynomial of degree $(k-1)$ in the variable $x(s)$. Therefore, $\dfrac{\triangle}{\triangle x(s)} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)$, $P_{n_1-1,n_2}^{\alpha_1+1,\alpha_2,\alpha_0+1,N-1}(s)$ and $P_{n_1,n_2-1}^{\alpha_1,\alpha_2+1,\alpha_0+1,N-1}(s)$ belongs to the vector space $\mathbb{P}_{n_1+n_2-1}$.

The previous lemma 2.2 enables us to find a explicit expression for the *lowering operator*. Indeed, the following lemma holds.

Lemma 2.3

$$\frac{\triangle}{\triangle x(s)} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s) \in Span\left\{P_{n_1-1,n_2}^{\alpha_1+1,\alpha_2,\alpha_0+1,N-1}(s), P_{n_1,n_2-1}^{\alpha_1,\alpha_2+1,\alpha_0+1,N-1}(s)\right\}.$$

Proof. It is sufficient to find constants λ_1 and λ_2, where $|\lambda_1|+|\lambda_2|>0$, such that $\dfrac{\triangle}{\triangle x(s)} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s) = \lambda_1 P_{n_1-1,n_2}^{\alpha_1+1,\alpha_2,\alpha_0+1,N-1}(s) + \lambda_2 P_{n_1,n_2-1}^{\alpha_1,\alpha_2+1,\alpha_0+1,N-1}(s)$, yields. Hence, for $k=1,2$ one has

$$\begin{aligned}
&\sum_{s=0}^{N}\left(\frac{\triangle}{\triangle x(s)} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)\right)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2})\\
&= \lambda_1\sum_{s=0}^{N} P_{n_1-1,n_2}^{\alpha_1+1,\alpha_2,\alpha_0+1,N-1}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2})\\
&+\lambda_2\sum_{s=0}^{N} P_{n_1,n_2-1}^{\alpha_1,\alpha_2+1,\alpha_0+1,N-1}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2}).
\end{aligned} \tag{16}$$

From lemma 2.2 expression (16) transforms into

$$\begin{aligned}
&\sum_{s=0}^{N}\left(\frac{\triangle}{\triangle x(s)} P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)\right)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2})\\
&= \left(\sum_{l=1}^{2}\lambda_l a_{k,l}\right)\sum_{s=0}^{N} P_{n_1-1,n_2-1}^{\alpha_1+1,\alpha_2+1,\alpha_0+2,N-2}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+2,N-2}(s)\triangle x(s-\tfrac{1}{2}).
\end{aligned} \tag{17}$$

By using summation by parts as well as the orthogonality property (10) one transforms the left hand side of the equation (17) as follows

$$
\begin{aligned}
&\sum_{s=0}^{N}\left(\frac{\triangle}{\triangle x(s)}P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)\right)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2})\\
&=\frac{[n_k+\alpha_0+\alpha_k+1]_q}{q^{\frac{N+n_k+\alpha_0+\alpha_k+1}{2}}}\sum_{s=0}^{N}P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0,N}(s)\triangle x(s-\tfrac{1}{2}).
\end{aligned}
$$

Using (15) one has

$$
\begin{aligned}
&\sum_{s=0}^{N}\left(\frac{\triangle}{\triangle x(s)}P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)\right)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2})\\
&\qquad=\frac{q^{\frac{n_1+n_2-1}{2}}[n_k+\alpha_0+\alpha_k+1]_q[n_k+\alpha_k-\alpha_l]_q}{[n_1+n_2+\alpha_l+\alpha_0+1]_q}\\
&\times\sum_{s=0}^{N}P_{\vec{n}-\vec{e}_l}^{\vec{\alpha}+\vec{e}_l,\alpha_0+1,N-1}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2}),\quad l=1,2.
\end{aligned}
$$

Applying again lemma 2.2 one gets

$$
\begin{aligned}
&\sum_{s=0}^{N}\left(\frac{\triangle}{\triangle x(s)}P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s)\right)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+1,N-1}(s)\triangle x(s-\tfrac{1}{2})\\
&\qquad=\frac{q^{\frac{n_1+n_2-1}{2}}[n_k+\alpha_0+\alpha_k+1]_q[n_k+\alpha_k-\alpha_l]_q}{[n_1+n_2+\alpha_l+\alpha_0+1]_q}\\
&\times a_{k,l}\sum_{s=0}^{N}P_{n_1-1,n_2-1}^{\alpha_1+1,\alpha_2+1,\alpha_0+2,N-2}(s)(s)_q^{[n_k-1]}\upsilon_q^{\alpha_k+1,\alpha_0+2,N-2}(s)\triangle x(s-\tfrac{1}{2}).
\end{aligned}
\tag{18}
$$

From equations (17) and (18) the equation (16) leads to the following linear system of equation for the unknowns λ_1 and λ_2,

$$
y_k=\lambda_1 a_{k,1}+\lambda_2 a_{k,2},\quad k=1,2,
\tag{19}
$$

being

$$
y_k=q^{\frac{N+n_k+\alpha_k+\alpha_0-2}{2}+n_1+n_2}\frac{[n_k+\alpha_0+\alpha_k+1]_q[n_k+\alpha_k-\alpha_1]_q[n_k+\alpha_k-\alpha_2]_q}{[n_1+n_2+\alpha_0+\alpha_1+1]_q[n_1+n_2+\alpha_0+\alpha_2+1]_q}.
$$

Notice that the explicit expression for the matrix coefficients of (19) is given in (13). Hence,

$$
\det\begin{pmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{pmatrix}=\frac{q^{N+2n_1+2n_2+2\alpha_0}\left(q^{\alpha_2}-q^{\alpha_1}\right)\left(q^{n_1+\alpha_1}-q^{n_2+\alpha_2}\right)}{\left(q^{1+n_1+n_2+\alpha_0+\alpha_1}-1\right)\left(q^{1+n_1+n_2+\alpha_0+\alpha_2}-1\right)}\neq 0.
$$

Since $y_1\neq y_2$ the statement holds. $\square$

Theorem 2.1 *The monic q-Hahn multiple orthogonal polynomial verify the following q-difference equation*

$$\left(\prod_{i=1}^{2} D^{\alpha_i+1,\alpha_0+1,N-1}\right)\frac{\triangle}{\triangle x(s)}P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s) = -q^{-\frac{(N+n_1+n_2+\alpha_0-1)}{2}}$$
$$\times\left(\frac{\lambda_1 D^{\alpha_2+1,\alpha_0+1,N-1}}{[n_1+n_2+\alpha_0+\alpha_1+1]_q^{-1}}+\frac{\lambda_2 D^{\alpha_1+1,\alpha_0+1,N-1}}{[n_1+n_2+\alpha_0+\alpha_2+1]_q^{-1}}\right)P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s), \tag{20}$$

where $\lambda_1 = \frac{x(n_1)x(n_2+\alpha_2-\alpha_1)}{x(\alpha_2-\alpha_1)}$, $\lambda_2 = \frac{x(n_2)x(n_1+\alpha_1-\alpha_2)}{x(\alpha_1-\alpha_2)}$.

Proof. From lemma 2.3 one has

$$\frac{\triangle}{\triangle x(s)}P_{n_1,n_2}^{\alpha_1,\alpha_2,\alpha_0,N}(s) = \lambda_1 P_{n_1-1,n_2}^{\alpha_1+1,\alpha_2,\alpha_0+1,N-1}(s) + \lambda_2 P_{n_1,n_2-1}^{\alpha_1,\alpha_2+1,\alpha_0+1,N-1}(s). \tag{21}$$

Notice that from system (19) $\lambda_1 = \frac{x(n_1)x(n_2+\alpha_2-\alpha_1)}{x(\alpha_2-\alpha_1)}$ and $\lambda_2 = \frac{x(n_2)x(n_1+\alpha_1-\alpha_2)}{x(\alpha_1-\alpha_2)}$.

Now, applying $\left(\prod_{i=1}^{2} D^{\alpha_i+1,\alpha_0+1,N-1}\right)$ on both sides of the equation (21) and considering that raising operators (11) are commuting then the expression (20) holds. □

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